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LOCALIZED FAMILIES OF BENDING WAVES IN A NON-CIRCULAR CYLINDRICAL SHELL WITH SLOPING EDGES†

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The initial-boundary-value problem for the equations of shallow shells describing the motion of a non-circular cylindrical shell is considered. The shell edges are given by not necessarily plane curves. The conditions of a joint support or a rigid clamp are considered as boundary conditions. It is assumed that the initial displacements and velocities of the points of the median surface of the shell are functions which decrease rapidly away from some generatrix. In the case when the shell edges lie in planes perpendicular to the generatrix, the solution of the problem can be constructed as an expansion in beam functions along the generatrix. The expansion enables the original initial boundary-value problem to be reduced to an initial problem, the solution of which can be constructed [1] by Maslov's method [2]. A complex WKB procedure is proposed, which is suitable for non-circular cylindrical shells with sloping edges. An asymptotic solution of the equations of motion is constructed by superimposing localized families (wave packets) of flexural waves travelling in a circular direction. A qualitative analysis of the solutions is carried out. As an example wave forms of motion of a cylindrical shell of oblique section are considered.

1. FORMULATION OF THE PROBLEM

On the median surface of a shell of thickness h we introduce an orthogonal system of coordinates s, φ, where s is the longitudinal coordinate and φ is a coordinate on the directrix chosen in such a way that $d\sigma^2 = R^2(ds^2 + d\varphi^2)$ is the first quadratic form of the surface. The radius of curvature is $R_2 = R/k(\varphi)$. Here R is a characteristic dimension of the median surface. Suppose that the shell is bounded by two edges and is not necessarily closed in the direction of φ

$$s_1(\varphi) \leqslant s \leqslant s_2(\varphi), \quad \varphi_1 \leqslant \varphi \leqslant \varphi_2$$

The functions $k(\varphi)$ and $s_i(\varphi)$ are assumed to be infinitely differentiable, with $\partial^m k/\partial\varphi^m$, $\partial^m s_i/\partial\varphi^m \sim 1$ as $\varepsilon \to 0$ $(m = 1, 2, \ldots)$.

Assuming that the waves vary rapidly with respect to the circular coordinate φ, we use the following system of equations [3] written in dimensionless form

$$\varepsilon^4\Delta^2 W + k(\varphi)\frac{\partial^2 F}{\partial s^2} + \varepsilon^2\frac{\partial^2 W}{\partial t^2} = 0, \quad \varepsilon^4\Delta^2 F - k(\varphi)\frac{\partial^2 W}{\partial s^2} = 0 \tag{1.1}$$

$$\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial\varphi^2}, \quad \varepsilon^8 = \frac{h^2}{12R^2(1-\nu^2)}, \quad t = \frac{t_*}{T_*}$$

$$W = \varepsilon^4\frac{W_*}{R}, \quad F = \varepsilon^{-4}\frac{F_*}{hE}, \quad T_*^2 = \varepsilon^{-6}\frac{R^2\rho}{E}$$

where W_* is the normal deflection, F_* is the stress function, t_* is the time, ρ is the density of the material, $0 < \varepsilon$ is a natural small parameter, E and ν are Young's modulus and Poisson's ratio, and T_* is the characteristic time.

On the shell edges $s = s_1(\varphi)$, $s = s_2(\varphi)$ we consider one of two groups of boundary conditions, namely, the joint support group or the rigid clamp group. Each of these groups includes six versions of the

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boundary conditions [4, 5]. The stress state of the shell consists of the basic stress state and the edge-effect integrals [6]. To study the basic stress state on each edge one only needs to satisfy two basic conditions. Apart from terms of order ε^2 these conditions have the form [4]

$$W = \partial^2 W/\partial s^2 = 0, \qquad W = \partial W/\partial s = 0 \tag{1.2}$$

for the joint support and rigid clamp groups, respectively.

Consider the initial conditions

$$W|_{t=0} = W_0^*(s,\varphi,\varepsilon)\Phi_0, \quad \dot{W}|_{t=0} = i\varepsilon^{-1}V_0^*(s,\varphi,\varepsilon)\Phi_0 \tag{1.3}$$

$$\Phi_0 = \Phi_0(\varphi,\varepsilon) = \exp\{i\varepsilon^{-1}(a_0\varphi + \tfrac{1}{2}b_0\varphi^2)\}, \quad \operatorname{Im} b_0 > 0$$

where $a_0 (a_0 \neq 0)$ is a real number and W_0^*, V_0^* are complex-valued functions such that

$$\partial^m W_0^* / \partial s^m, \quad \partial^m V_0^* / \partial s^m \sim \varepsilon^{-m\gamma} \quad \text{when } \varepsilon \to 0 \tag{1.4}$$

$$m = 1,2,\ldots; \quad 0 \leqslant \gamma < 3/4; \quad 0 \leqslant \alpha \leqslant \tfrac{1}{2}$$

having a finite number of oscillations with variability of order $\varepsilon^{-\alpha}$ in the direction of φ. Conditions (1.3) specify an initial wave packet on the shell surface with variability of order ε^{-1} in the direction of φ localized in the neighbourhood of the generatrix $\varphi = 0$.

2. METHOD OF SOLUTION

Consider the equation

$$d^4 z/ds^4 - \lambda z = 0 \tag{2.1}$$

We will denote by $z_1(s, \varphi), z_2(s, \varphi), \ldots$ an infinite system of eigenfunctions of the boundary-value problem (1.2), (2.1). Suppose that W_0^*, V_0^* satisfy one version of the boundary conditions (1.2). Then for any $\varphi \in [\varphi_1, \varphi_2]$, W_0^* and V_0^* can be expanded in terms of the eigenfunctions $z_n(s, \varphi)$ into uniformly convergent series in the section $[\varphi_1, \varphi_2]$ [7]

$$W_0^* = \sum_{n=1}^{\infty} W_{n0}(\varphi,\varepsilon) z_n(s,\varphi), \quad W_{n0} = \int_{s_1(\varphi)}^{s_2(\varphi)} W_0^* z_n ds \tag{2.2}$$

$$V_0^* = \sum_{n=1}^{\infty} V_{n0}(\varphi,\varepsilon) z_n(s,\varphi), \quad V_{n0} = \int_{s_1(\varphi)}^{s_2(\varphi)} V_0^* z_n ds$$

Taking (1.4) and (2.2) into account, in practical computations it is possible to restrict oneself to a finite number $N \sim \varepsilon^{-\gamma}$ of terms. Let W_{n0}, V_{n0} be polynomials of $\varepsilon^{-1/2}\varphi$ whose coefficients are regular functions of ε. This assumption involves the presence of a finite number of oscillations in the amplitude of the initial wave packet [2]. Then W_{n0}, V_{n0} can be represented by the series

$$W_{n0} = \sum_{m=0}^{\infty} \varepsilon^{m/2} w_{nm}^0(\zeta), \quad V_{n0} = \sum_{m=0}^{\infty} \varepsilon^{m/2} v_{nm}^0(\zeta); \quad \zeta = \varepsilon^{-1/2}\varphi$$

where w_{nm}^0, v_{nm}^0 are polynomials of degree M_{nm} with (in general) complex coefficients. We take the Taylor expansion of z_n

$$z_n = z_n^0 + \sum_{r=1}^{\infty} \varepsilon^{r/2}\zeta^r \frac{\partial^r z_n^0}{\partial\varphi^r}; \quad z_n^0 = z_n(s,0), \quad \frac{\partial^r z_n^0}{\partial\varphi^r} = \left.\frac{\partial^r z_n}{\partial\varphi^r}\right|_{\varphi=0}$$

Following [1] and taking into account that the original system is linear, we shall seek a solution of problem (1.1)–(1.3), (2.2) in the form

$$W = \sum_{n=1}^{N} W_n, \quad F = \sum_{n=1}^{N} F_n \tag{2.3}$$

where W_n, F_n $(n = 1, 2, \dots, N)$ are the required functions, which at time t are localized in the neighbourhood of a generatrix $\varphi = q_n(t)$ and satisfy the initial conditions

$$W_n|_{t=0} = \sum_{m=0}^{\infty} \varepsilon^{m/2} w_{nm}^0 z_n, \quad \dot{W}_n|_{t=0} = i\varepsilon^{-1} \sum_{m=0}^{\infty} \varepsilon^{m/2} v_{nm}^0 z_n \tag{2.4}$$

where the functions $z_n = z_n(s, \varphi)$ are represented by the above Taylor series.

The pair W_n, F_n will be called the nth wave packet with centre at $\varphi = q_n(t)$. Here $q_n(t)$ is a twice differentiable function such that

$$q_n(0) = 0 \tag{2.5}$$

In (1.1) we change to a new system of coordinates connected with centre $q_n(t)$ using the formula

$$\varphi = q_n(t) + \varepsilon^{1/2} \xi_n \tag{2.6}$$

As a result, we obtain the system of equations

$$\varepsilon^2 \frac{\partial^4 W_n}{\partial \xi_n^4} + 2\varepsilon^3 \frac{\partial^4 W_n}{\partial \xi_n^2 \partial s^2} + \varepsilon^4 \frac{\partial^4 W_n}{\partial s^4} + k \frac{\partial^2 F_n}{\partial s^2} + \varepsilon^2 \frac{\partial^2 W_n}{\partial t^2} - 2\varepsilon^{3/2} \dot{q}_n \frac{\partial^2 W_n}{\partial \xi_n \partial t} +$$

$$+ \varepsilon \dot{q}_n^2 \frac{\partial^2 W_n}{\partial \xi_n^2} - \varepsilon^{3/2} \ddot{q}_n \frac{\partial W_n}{\partial \xi_n} = 0 \tag{2.7}$$

$$\varepsilon^2 \frac{\partial^4 F_n}{\partial \xi_n^4} + 2\varepsilon^3 \frac{\partial^4 F_n}{\partial \xi_n^2 \partial s^2} + \varepsilon^4 \frac{\partial^4 F_n}{\partial s^4} - k \frac{\partial^2 W_n}{\partial s^2} = 0$$

describing the behaviour of the nth wave packet.

We shall seek a solution of (2.7) with initial conditions (2.4) in the form

$$W_n = W_n^* \Phi_n, \quad F_n = F_n^* \Phi_n \tag{2.8}$$

$$W_n^* = \sum_{m=0}^{\infty} \varepsilon^{m/2} w_{nm}(s, \xi_n, t), \quad F_n^* = \sum_{m=0}^{\infty} \varepsilon^{m/2} f_{nm}(s, \xi_n, t)$$

$$\Phi_n = \exp\left\{ i \left[\varepsilon^{-1} \int_0^t \omega_n(\tau) d\tau + \varepsilon^{-1/2} p_n(t) \xi_n + \frac{1}{2} b_n(t) \xi_n^2 \right] \right\}$$

where ω_n, p_n, b_n are twice differentiable with respect to t, $\operatorname{Im} b_n(t) > 0$ for any $t > 0$, and w_{nm}, f_{nm} are polynomials in ξ_n. Here $\omega_n(t)$ is the instantaneous frequency of shell vibrations in the neighbourhood of the centre $\varphi = q_n(t)$, $p_n(t)$ determines the variability in the direction of ψ, and $b_n(t)$ characterizes the rate of decay of the wave amplitude as the distance from the centre $\varphi = q_n(t)$ increases.

Note that when $q_n = 0$ and ω_n, p_n, b_n are constants, expansions of the form (2.8) were constructed [8, 9] for the equations governing the stability and characteristic vibrations of shells.

We substitute (2.8) into (2.7) and expand $k(\varphi)$ in a Taylor series in powers of $\varepsilon^{1/2}\xi_n$ in the neighbourhood of the stationary point $\varphi = q_n(t)$. Equating the coefficients of like powers of $\varepsilon^{1/2}$ to zero and eliminating f_{nm}, we obtain the sequence of differential equations

$$\sum_{j=0}^{m} L_{nj} w_{nm-j} = 0, \quad m = 0, 1, 2, \dots \tag{2.9}$$

from which to determine $\omega_n, q_n, p_n, b_n, w_{nm}$. Here

$$L_{n0} = \frac{k^2[q_n(t)]}{p_n^4(t)}\frac{\partial^4}{\partial s^4} + \{p_n^4(t) - [\omega_n(t) - \dot{q}_n(t)p_n(t)]^2\}$$

$$L_{n1} = (b_n L_p + L_q + \dot{p}_n L_\omega)\xi_n - iL_p \frac{\partial}{\partial \xi_n} \tag{2.10}$$

$$L_{n2} = \frac{1}{2}(b_n^2 L_{pp} + 2b_n L_{pq} + L_{qq} + \dot{p}_n^2 L_{\omega\omega} + 2\dot{p}_n L_{\omega q} + 2\dot{p}_n b_n L_{\omega p} + \dot{b}_n L_\omega)\xi_n^2 +$$

$$+a_{n0}\frac{\partial^2}{\partial \xi_n^2} + a_{n1}\frac{\partial}{\partial \xi_n}\xi_n + a_{n2}\frac{\partial}{\partial t} + a_{n3}$$

$$a_{n0} = -\frac{1}{2}L_{pp}, \quad a_{n1} = -i(b_n L_{pp} + L_{pq} + \dot{p}_n L_{\omega p}), \quad a_{n2} = -iL_\omega$$

$$a_{n3} = -i\left(\frac{1}{2}b_n L_{pp} + \frac{1}{2}\dot{\omega}_n L_{\omega\omega} + \dot{p}_n L_{\omega p} - \frac{4kk'}{p_n^5}\frac{\partial^4}{\partial s^4} + \ddot{q}_n p_n\right)$$

The subscripts p, q, ω in (2.10) and below denote differentiation with respect to the corresponding variables.

The functions f_{nm} are found one after another from the inhomogeneous equations and can be expressed in terms of $f_{n0}, f_{n1}, \ldots, f_{nm-1}$. In particular, $f_{n0} = k(q_n)p_n^{-4}\partial^2 w_{n0}/\partial s^2$.

Substituting (2.8) into (1.2) we obtain a sequence of boundary conditions for w_{nm} with $s = s_i[q_n(t)]$. For example, in the case of a joint support we obtain

$$w_{n0} = 0, \quad \frac{\partial^2 w_{n0}}{\partial s^2} = 0 \tag{2.11}$$

$$w_{n1} + \xi_n s' \frac{\partial w_{n0}}{\partial s} = 0, \quad \frac{\partial^2 w_{n1}}{\partial s^2} + \xi_n s' \frac{\partial^3 w_{n0}}{\partial s^3} = 0 \tag{2.12}$$

$$w_{n2} + \xi_n s' \frac{\partial w_{n1}}{\partial s} + \frac{1}{2}\xi_n^2\left[s''\frac{\partial^2 w_{n0}}{\partial s} + s'^2\frac{\partial^3 w_{n0}}{\partial s^2}\right] = 0 \tag{2.13}$$

$$\frac{\partial^2 w_{n2}}{\partial s^2} + \xi_n s' \frac{\partial^3 w_{n1}}{\partial s^3} + \frac{1}{2}\xi_n^2\left[s''\frac{\partial^3 w_{n0}}{\partial s^3} + s'^2\frac{\partial^4 w_{n0}}{\partial s^4}\right] = 0$$

3. INTEGRATION OF EQUATIONS (2.9)

Consider the boundary-value problem (2.9), (2.11) which arises in the null approximation ($m = 0$). We shall seek its solution in the form

$$w_{n0} = P_{n0}(\xi_n, t)z_n[s, q_n(t)] \tag{3.1}$$

where $P_{n0}(\xi_n, t)$ is a polynomial of argument ξ_n. Substituting (3.1) into (2.9) for $m = 0$ we obtain

$$\omega_n(t) = \dot{q}_n(t)p_n(t) \mp H_n[p_n(t), q_n(t)]$$

$$H_n(p_n, q_n) = [p_n^4 + \lambda_n(q_n)k^2(q_n)p_n^{-4}]^{1/2} \tag{3.2}$$

where $H_n(p_n, q_n)$ is the Hamilton function and $\lambda_n[q_n(t)]$ is an eigenvalue of the boundary-value problem (2.1), (2.11) for $s = s_i[q_n(t)]$.

In the first approximation ($m = 1$) we have the boundary-value problem (2.9), (2.12) for w_{n1}. We shall seek a solution of the latter in the form

$$w_{n1} = P_{n1}(\xi_n, t) z_n[s, q_n(t)] + w_{n1}^{(p)}(s, \xi_n, t) \tag{3.3}$$

where P_{n1} is a polynomial of argument ξ_n and $w_{n1}^{(p)}$ is a partial solution of (2.9) for $m = 1$. The equality

$$\int_{s_1}^{s_2} z_n(L_{n0} w_{n1} + L_{n1} P_{n0} z_n) ds = 0 \tag{3.4}$$

serves as a condition for the existence of w_{n1}. It is a differential equation in P_{n0}. For the latter to have a solution in the form of a polynomial of argument ξ_n it is necessary that $p_n(t)$, $q_n(t)$ should satisfy identically the Hamilton system

$$\dot{q}_n = H_p, \quad \dot{p}_n = -H_q \tag{3.5}$$

Let $p_n(t)$, $q_n(t)$ be a solution of (3.5) with initial conditions $p_n(0) = a_0$, $q_n(0) = 0$. Then $w_{n1}^{(p)} = \xi_n P_{n0} z_q$, where $z_q = \partial z_n/\partial q_n$. In this approximation the polynomials P_{n0}, P_{n1} remain undefined.

Considered (2.9) for $m = 2$ with boundary conditions (2.13). Taking (3.2) and (3.5) into account, the condition for a solution of this problem to exist leads to the equation

$$(\xi_n^2 D_b - 2D_{\xi t}) P_{n0} = 0 \tag{3.6}$$

for P_{n0}. Here

$$D_b = \dot{b}_n + H_{pp} b_n^2 + 2H_{pq} b_n + H_{qq}$$

$$D_{\xi t} = a_{n0}^* \frac{\partial^2}{\partial \xi_n^2} + a_{n1}^* \xi_n \frac{\partial}{\partial \xi_n} + a_{n2}^* \frac{\partial}{\partial t} + a_{n3}^*$$

$$a_{n0}^*(t) = \tfrac{1}{2} H_{pp}, \quad a_{n1}^*(t) = i(b_n H_{pp} + H_{pq}), \quad a_{n2}^* = i$$

$$a_{n3}^*(t) = i\eta_n^{-1}(2H_n)^{-1}[H_n H_{pp} b_n - \dot{\omega}_n - 2H_p H_q - 4\lambda_n(q_n) k(q_n) k'(q_n) p_n^{-5} + \ddot{q}_n p_n +$$

$$+ \int_{s_1}^{s_2} (L_n z_q + L_\omega \dot{z}_n) z_n ds, \quad \eta_n(t) = \int_{s_1}^{s_2} z_n^2 ds$$

For (3.6) to have a solution in the form of a polynomial it is necessary that $b_n(t)$ be a solution of the Riccati equation

$$\dot{b}_n + H_{pp} b_n^2 + 2H_{pq} b_n + H_{qq} = 0 \tag{3.7}$$

Let $b_n(t)$ be a solution of (3.7) that satisfies the initial condition $b_n(0) = b_0$. It can be proved [10, p. 104] that if $\operatorname{Im} b_0 > 0$, then $0 < \operatorname{Im} b_n(t) < +\infty$ in any finite interval $0 < t < T$.

Using (3.7), Eq. (3.6) takes the form $D_{\xi t} P_{n0} = 0$. Any polynomial

$$P_{n0}(\xi_n, t) = \sum_{k=0}^{M} A_{nk}(t) \xi_n^k \tag{3.8}$$

of degree M with coefficients

$$A_{nM}(t) = d_{n0} \Psi_{n0}(t), \quad A_{nM-1}(t) = d_{n1} \Psi_{n1}(t)$$

$$A_{nM-r}(t) = \Psi_{nr}(t) \left[d_{nr} - (M - r + 2)(M - r + 1) \int \frac{a_{n0}^*(t) A_{nM-r+2}(t)}{a_{n2}^*(t) \Psi_{nr}(t)} dt \right]$$

$$\Psi_{nj}(t) = \exp\left\{ -\int \frac{(M - j) a_{n1}^*(t) + a_{n3}^*(t)}{a_{n2}^*(t)} dt \right\} \tag{3.9}$$

$$r = 2, 3, \ldots, M; \; j = 0, 1, \ldots, M$$

is a solution of this equation. Here d_{nj} are arbitrary complex numbers, which can be determined from the initial conditions of the problem.

The function $W_n = [w_{n0} + O(\varepsilon^{1/2})]\Phi_n$ found from the first three approximations is the leading term in the asymptotic expansion of the solution (2.8) and satisfies the original boundary conditions (1.2) apart from terms $O(\varepsilon^{1/2})$. To determine the correction $\varepsilon^{m/2}w_{nm}$ in (2.8) for $m \geqslant 1$ one must consider the corresponding boundary-value problem in the $(m + 2)$nd approximation. The existence of a solution of the latter leads to the inhomogeneous differential equation $D_{\xi,t}P_{nm} = P^*$ for the polynomial $P_{nm}(\xi_n, t)$. Note that the above procedure for constructing the polynomials w_{nm} is no longer valid for $m \geqslant 4$ because the correction introduced by the boundary-value problem into the general solution (2.8) at the sixth step is of order $O(\varepsilon^2)$ at the shell edges, which is the same as the error of the original boundary conditions (1.2).

4. DETERMINATION OF THE CONSTANTS d_{nj}

Taking (3.2) into account, we denote by $p_n^\pm$, $q_n^\pm$, $\omega_n^\pm$, $b_n^\pm$, $z_n^\pm$, $P_n^\pm$, $w_n^\pm$ the positive and negative branches of the functions found above, corresponding to the Hamiltonians H_n and $-H_n$. Here $z_n^\pm = z_n[s, q_n^\pm(t)]$. Let $\xi_n^\pm = \varepsilon^{-1/2}[\varphi - q_n^\pm(t)]$. Then $P_{n0}^\pm$ are polynomials of argument $\xi_n^\pm$ containing the undetermined constants $a_{nj}^\pm$. We consider the functions

$$W_n = W_n^+ + W_n^-, \quad F_n = F_n^+ + F_n^- \tag{4.1}$$

$$W_n^\pm = [w_{n0}^\pm + O(\varepsilon^{1/2})]\Phi_n^\pm, \quad F_n^\pm = [f_{n0}^\pm + O(\varepsilon^{1/2})]\Phi_n^\pm$$

where the plus and minus superscripts indicate that the computations are carried out for the positive and negative branches, respectively. By the above construction, functions (4.1) satisfy Eqs (2.7) in the leading approximation. To determine the constants $d_{nj}^\pm$ appearing in W_n and F_n we substitute (4.1) into initial conditions (2.4) and use the equality $\xi_n^\pm = \zeta$ and the identity $z_n^\pm \equiv z_n^0$, which hold at $t = 0$. As a result, we obtain the system of equations

$$P_n^\pm|_{t=0} = \frac{1}{2}\left[w_{n0}^0(\zeta) \mp \frac{v_{n0}^0(\zeta)}{H_n^0}\right], \quad H_n^0 = H_n(a_0, 0) \tag{4.2}$$

from which to determine $d_{nj}^\pm$. From (4.2) it follows that the polynomials $P_{n0}^\pm$ have degree $M = M_{n0}$.

5. ANALYSIS OF THE SOLUTION

When k and s_i are constants the functions (2.3) and (4.1) are identical with the solution found in [1] by Maslov's method [2].

The terms with plus and minus superscripts in (4.1) will be called the n^+th and n^-th wave packets, respectively. An analysis of (2.3) and (4.1) shows that $|W_n| = O(\varepsilon^\infty)$ outside the neighbourhoods of the generating lines $\varphi = q_n^\pm(t)$ when n and t are fixed. This means that the initial wave packet (1.3) splits into $2N$ packets for $t > 0$, the n^+th and n^-th packets moving in opposite directions to the generatrix $\varphi = 0$ with group velocities $v_{ng}^\pm = \dot{q}_n^\pm(t)$. The width of the packets is of order $\varepsilon^{1/2}/\text{Im}\, b_n^\pm(t)$.

The behaviour of the wave packets depends strongly on $k(\varphi)$, $s_i(\varphi)$. In (3.2) we introduce the symbol $g_n = \lambda_n(q_n)k^2(q_n)$. Let us consider the following cases.

1. $g'_n(\varphi) < 0$ for $0 \leqslant \varphi \leqslant \varphi_2$. From an analysis of (3.5) we find that for any $t > 0$ we have

$$\dot{p}_n^+ > 0, \quad v_{ng}^+ > 0, \quad \dot{v}_{ng}^+ > 0, \quad \text{if} \quad a_0^8 \geqslant g_n(0)$$

$$\dot{p}_n^- < 0, \quad v_{ng}^- > 0, \quad \dot{v}_{ng}^- > 0, \quad \text{if} \quad 0 < a_0^8 < g_n(0)$$

The latter shows that one of the $n^\pm$th packets moves in the direction of decreasing $g_n(\varphi)$ with increasing group velocity.

2. $g'_n(\varphi) > 0$ for $0 \leqslant \varphi \leqslant \varphi_2$. Let $H_n^\circ > (4K_n)^{1/4}$, where $K_n = \sup g_n(\varphi)$ on the set $0 \leqslant \varphi \leqslant \varphi_2$. Here

$$\dot{p}_n^+ < 0, \quad v_{ng}^+ > 0, \quad \dot{v}_{ng}^+ < 0 \quad \text{when} \quad a_0^8 \geqslant g_n(0)$$

$$\dot{p}_n^- > 0, \quad v_{ng}^- > 0, \quad \dot{v}_{ng}^- < 0 \quad \text{when} \quad 0 < a_0^8 < g_n(0)$$

In this case one of the n^+th packets moves in the direction of increasing values of $g_n(\varphi)$, but its group velocity decreases.

Now let

$$\overset{\circ}{H}_n \leqslant (4K_n)^{1/4} \tag{5.1}$$

Here for $a_0^8 > g_n(0)$ a $t_r^+ > 0$ exists such that

$$\begin{aligned} &\dot{p}_n^+ < 0, \quad v_{ng}^+ > 0, \quad \dot{v}_{ng}^+ < 0 \quad \text{for} \quad 0 < t < t_r^+ \\ &v_{ng}^+ = 0 \quad \text{for} \quad t = t_r^+ \\ &\dot{p}_n^+ < 0, \quad v_{ng}^+ < 0, \quad \dot{v}_{ng}^+ < 0 \quad \text{for} \quad t > t_r^+ \end{aligned} \tag{5.2}$$

If $0 < a_0^8 < g_n(0)$, then there is $t_r^- > 0$ such that relations similar to (5.2) hold when the plus superscript is replaced by a minus and the inequality for $\dot{p}_n^-$ is reversed. Thus, if (5.1) is satisfied, one of the $n^\pm$th packets is reflected from a certain generatrix $\varphi^\pm_{nr} = q_n^\pm(t_r^\pm)$, which can be determined from the equation

$$\overset{\circ}{H}_n = [4\lambda_n(\varphi)k^2(\varphi)]^{1/4}$$

Finally, when $a_0^8 = g_n(0)$, subject to condition (5.1), it is impossible for both $n^\pm$th packets to move in the direction of increasing $g_n(\varphi)$.

3. The case when the "weakest" [8] generatrix $\varphi = 0$, for which $g_n'(0) = 0, g_n''(0) > 0$, exists on the shell surface is of particular interest. The following forms of free vibrations with the lowest frequency $\omega_n^w = H_n^w + \varepsilon\chi/2$ are localized in a neighbourhood of this generatrix

$$W = z_n^0 \exp\{i\varepsilon^{-1}[\omega_n^w t + p_n^w\varphi + \tfrac{1}{2}b_n^w\varphi^2]\} \tag{5.3}$$

where n is the number of half-wavelengths along the generatrix. Here $\chi = [H_{pp}^w H_{qq}^w - (H_{pq}^w)^2]^{1/2}$. The superscript w indicates that the values of H_n and its derivatives are taken for $p = p_n^w = g_n^{1/8}(0)$ and $q = q_n^w = 0$ (on the "weakest" generatrix). Note that p_n^w, q_n^w, b_n^w can be found from Eqs (3.5) and (3.7), respectively, in which one must take $\dot{p}_n$, $\dot{q}_n$, $\dot{b}_n$ to be identically equal to zero.

If $a_0 \neq p_n^w$ and (5.1) is satisfied, then the $n^\pm$th packets undergo oscillatory motion about the "weakest" generatrix, being repeatedly reflected from the line $\varphi = \varphi_{nr}^\pm$.

Now let $a_0 = p_n^w$. Then from (3.5) we obtain $p_n = p_n^w$, $q_n = 0$ for any $t \geqslant 0$, which demonstrates that no splitting of the initial nth packet (2.4) occurs. If $b_0 \neq b_n^w$, then $b_n(t)$ is a function of time, and if $b_0 = b_n^w$, we obtain $b_n = b_n^w$ for any $t \geqslant 0$. In the latter case the nth package undergoes motions identical, apart from amplitude, with the characteristic form (5.3) of shell vibrations.

Therefore the presence of the "weakest" generatrix can lead to the localization of wave forms of shell motion in the neighbourhood of this generatrix.

6. EXAMPLE

We consider a joint-supported circular cylindrical shell with a sloping edge. Let

$$k = 1, \quad s_1 = 0, \quad l = s_2(\varphi) = l_0 + \operatorname{tg}\beta\cos\varphi$$

$$W_0^* = \overset{\circ}{w}_{n0}\sin(\pi ns/l), \quad V_0^* = \overset{\circ}{v}_{n0}\sin(\pi ns/l)$$

where β is the angle of inclination of the edge, n is a natural number, and l_0, w_{n0}°, v_{n0}° are constants. Then $\lambda_n(\varphi) = (\pi n/l)4$ and $z_n(s, \varphi) = \sin(\pi ns/l)$. In the case in question the initial wave packet is concentrated on the "weakest" generatrix $\varphi = 0$ of length $l_0 + \operatorname{tg}\beta$. Computations have been carried out for $h/R = 4 \times 10^{-3}$, $l_0 = 1$, $R = 50$ cm, $\nu = 0.3$, $E = 6.24 \times 10^{-7}$ kg/(cm s^2), $\rho = 1.18 \times 10^{-3}$ kg/cm^3, $a_0 = 2$, $b_0 = 3$, $n = 1$, $\beta = 30°$, $w_{n0}^\circ = 1$, $v_{n0}^\circ = 0$. Inequalities (5.1) hold and $a_0^8 > g_1(0)$ for the parameter values under consideration.

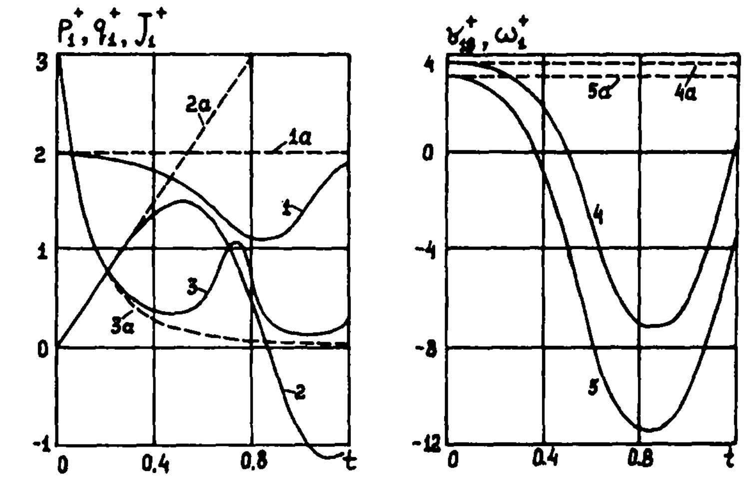

Fig. 1.

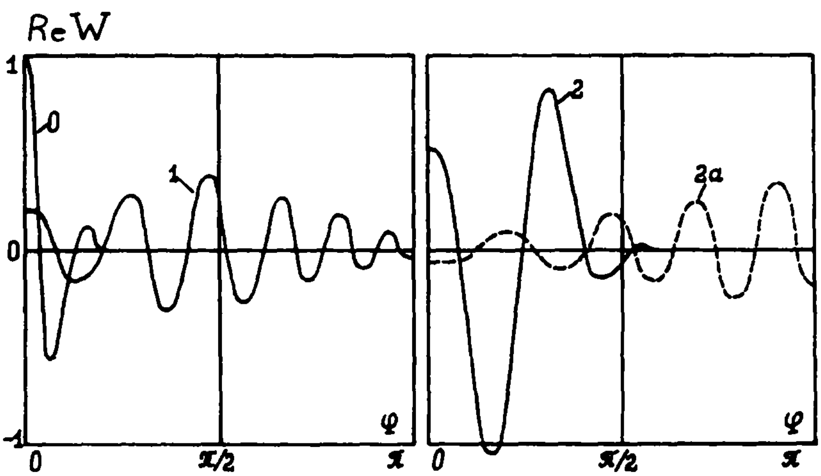

Fig. 2.

In Fig. 1 the functions $p_1^+(t)$, $q_1^+(t)$, $J_1^+(t)$ = Im $b_1^+(t)$, $v_{1g}^+(t)$, $\omega_1^+(t)$ are marked by the numbers 1–5. For comparison, the dashed lines 1a–5a show the same functions for a shell with a straight edge $s_2 = l_0 + \text{tg } 30°$. Curves 1–5 indicate that the behaviour of the 1^+th packet is entirely consistent with the qualitative analysis of the solution obtained earlier: first the packet moves in the direction of decreasing generatrix length, then it spreads out, and then it is reflected from the generatrix $\varphi_{1r}^+ = 1.45$ at time $t = 0.51$, followed by focusing.

The wave pattern over the section $s = l(\varphi)/2$ of the shell surface is shown in Fig. 2. The numbers 0–2 indicate the waves at $t = 0$, $t = 0.4$ (before the packet is reflected), and $t = 0.75$ (at the time of focusing after reflection), respectively. The dashed line 2a represents the solution at $t = 0.75$ for a shell with a straight edge $s_2 = l_0 + \text{tg } 30°$. It can be seen that the presence of a sloping edge increases the amplitude of the reflected wave.

The error of the method proposed here depends very much on the relations between the input parameters of the problem. In particular, if the shell has a sloping edge, the error increases as the angle of inclination β increases and/or the number b_0 decreases. This is because the solution (2.3), (4.1) satisfies the boundary conditions on the sloping edge precisely only on the generating lines $\varphi = q_n^{\pm}(t)$ and approximately with accuracy $O(\varepsilon^{1/2})$ away from them. The inaccuracy in satisfying the boundary conditions on the sloping edge leads to an accumulation of errors as the wave amplitude is computed (see (3.9)) for large t.

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